EQUILIBRIUM AND PARETO EFFICIENCY

Environment:

Pure exchange economy with two infinitely lived consumers and one good per period.

Utility: $\sum_{t=0}^{\infty} \beta_i^t \log c_t^i$ where $0 < \beta_i < 1$, $i = 1,2$.

Endowments: $(w_0^i, w_1^i, w_2^i, ...)$ where $w_t^i > 0$, $i = 1,2$, $t = 0,1,2,...$.

Market structure:

With an Arrow-Debreu markets structure, futures markets for goods are open in period 0. Consumers trade futures contracts among themselves.

Equilibrium:

An **Arrow-Debreu equilibrium** is a sequence of prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$ and an allocation $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, ...;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, ...$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, consumer *i*, $i = 1,2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, ...$ to solve

$$\begin{aligned} \max\ & \sum_{t=0}^{\infty} \beta_i^t \log c_t^i \\ \text{s.t. } & \sum_{t=0}^{\infty} \hat{p}_t c_t^i \le \sum_{t=0}^{\infty} \hat{p}_t w_t^i \\ & c_t^i \ge 0. \end{aligned}$$

- $\hat{c}_t^1 + \hat{c}_t^2 \le w_t^1 + w_t^2$, $=$ if $\hat{p}_t > 0$, $t = 0,1,2,\ldots$.

Characterization of equilibrium using calculus:

The Kuhn-Tucker theorem says that $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, ...$ solves the consumer's maximization problem if and only if there exists a Lagrange multiplier $\hat{\lambda}_i \ge 0$ such that

$$\beta_i^t \frac{1}{\hat{c}_t^i} - \hat{\lambda}_i \hat{p}_t \le 0,\ = 0 \text{ if } \hat{c}_t^i > 0$$

$$\sum_{t=0}^{\infty} \hat{p}_t w_t^i - \sum_{t=0}^{\infty} \hat{p}_t c_t^i \geq 0, \ = 0 \text{ if } \hat{\lambda}^i > 0.$$

For any t, $t = 0,1,2,\ldots$, $\lim_{c \to 0} \beta_i^t \frac{1}{c} = \infty$ implies that $\hat{c}_t^i > 0$, which implies that $\hat{\lambda}_i > 0$. It also implies that $\hat{p}_t > 0$, $t = 0,1,2,\ldots$. Consequently, $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots; \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;$ $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an equilibrium if and only if there exist Lagrange multipliers $\hat{\lambda}^1, \hat{\lambda}^2$, $\hat{\lambda}^i > 0$, such that

- $\beta_i^t \frac{1}{\hat{c}_t^i} = \hat{\lambda}_i \hat{p}_t$, $i = 1,2$, $t = 0,1,2,\ldots$
- $\sum_{t=0}^{\infty} \hat{p}_t c_t^i = \sum_{t=0}^{\infty} \hat{p}_t w_t^i$, $i = 1,2$
- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,2,\ldots$.

Pareto efficiency:

An allocation $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is **Pareto efficient** if it is feasible,

$$\hat{c}_t^1 + \hat{c}_t^2 \leq w_t^1 + w_t^2, \ t = 0,1,2,\ldots,$$

and there exists no other allocation, $\bar{c}_0^1, \bar{c}_1^1, \bar{c}_2^1, \ldots;\ \bar{c}_0^2, \bar{c}_1^2, \bar{c}_2^2, \ldots$ that is also feasible and is such that

$$\sum_{t=0}^{\infty} \beta_i^t \log \bar{c}_t^i > \sum_{t=0}^{\infty} \beta_i^t \log \hat{c}_t^i, \text{ some } i,\ i = 1,2, \text{ and}$$

$$\sum_{t=0}^{\infty} \beta_i^t \log \bar{c}_t^i \geq \sum_{t=0}^{\infty} \beta_i^t \log \hat{c}_t^i, \text{ all } i,\ i = 1,2.$$

Alternatively,

An allocation $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is **Pareto efficient** if and only if there exist numbers $\hat{\alpha}_1, \hat{\alpha}_2$, $\hat{\alpha}_i \geq 0$ and not both 0, such that $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ solves

$$\begin{aligned} \max\ & \hat{\alpha}_1 \sum_{t=0}^{\infty} \beta_1^t \log c_t^1 + \hat{\alpha}_2 \sum_{t=0}^{\infty} \beta_2^t \log c_t^2 \\ \text{s.t}\ & c_t^1 + c_t^2 \leq w_t^1 + w_t^2,\ t = 0,1,2,\ldots \\ & c_t^i \geq 0. \end{aligned}$$

(Note: It is easy to show that, if an allocation solves the above social planner's problem, it satisfies the first definition of Pareto efficiency. It is a little more difficult to show that, if an allocation satisfies the first definition of Pareto efficiency, there exist welfare weights $\hat{\alpha}_1, \hat{\alpha}_2$ such that the allocation solves the social planner's problem.)

Characterization of Pareto efficiency using calculus:

The Kuhn-Tucker theorem says that $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, ...;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, ...$ solves the social planner's problem if and only if there exists a Lagrange multipliers $\hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, ...,\ \hat{\pi}_t \geq 0$, such that

$$\hat{\alpha}_i \beta_i^t \frac{1}{\hat{c}_t^i} - \hat{\pi}_t \leq 0,\ = 0 \text{ if } \hat{c}_t^i > 0$$

$$w_t^1 + w_t^2 - \hat{c}_t^1 + \hat{c}_t^2 \geq 0,\ = 0 \text{ if } \hat{\pi}_t > 0.$$

For any t, $t = 0,1,2,\ldots$, $\lim_{c \to 0} \beta_i^t \frac{1}{c} = \infty$ implies that $\hat{c}_t^i > 0$, which implies that $\hat{\pi}_t > 0$. Consequently, $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, ...;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, ...$ is a Pareto efficient allocation if and only if there exist Lagrange multipliers $\hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, ...,\ \hat{\pi}_t > 0$, such that

- $\hat{\alpha}_i \beta_i^t \frac{1}{\hat{c}_t^i} = \hat{\pi}_t$, $i = 1,2$, $t = 0,1,2,\ldots$
- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0,1,2,\ldots$.

(Note: Since $\hat{\alpha}_i > 0$ for at least one *i*, $i = 1,2$, we know that, for that consumer i, $\hat{c}_t^i > 0$ for all t, $t = 0,1,2,\ldots$, and, consequently, that $\hat{\pi}_t > 0$. If one of the welfare weights $\hat{\alpha}_i$ equals 0, then $\hat{c}_t^i = 0$. We can imagine the first order conditions for that consumer i as being satisfied in the limit or we can simply ignore them. In what follows, we avoid the case where one of the welfare weights equals 0.)

First welfare theorem:

Suppose that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, ...;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, ...$ is an equilibrium. Then the allocation $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, ...;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, ...$ is Pareto efficient.

Proof:

Since $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is an equilibrium, we know that there exist Lagrange multipliers $\hat{\lambda}_1, \hat{\lambda}_2,\ \hat{\lambda}_i > 0$, such that

$$\beta_i^t \frac{1}{\hat{c}_t^i} = \hat{\lambda}_i \hat{p}_t$$

$$\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$$

We also know that, if there exist welfare weights $\hat{\alpha}_1, \hat{\alpha}_2,\ \hat{\alpha}_i > 0$, and Lagrange multipliers $\hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, \ldots,\ \hat{\pi}_t > 0$, such that

$$\hat{\alpha}_i \beta_i^t \frac{1}{\hat{c}_t^i} = \hat{\pi}_t$$

$$\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2,$$

then $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is a Pareto efficient allocation. (In other words, we are given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ and $\hat{\lambda}^1, \hat{\lambda}^2$ that satisfy certain properties, and we want to construct $\hat{\alpha}_1, \hat{\alpha}_2$ and $\hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, \ldots$ that, together with $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;$ $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$, satisfy certain other properties.) To prove the theorem, we set

$$\hat{\alpha}_i = \frac{1}{\hat{\lambda}_i}$$

$$\hat{\pi}_t = \hat{p}_t. \qquad \blacksquare$$

Equilibrium with transfers:

An **Arrow-Debreu equilibrium with transfers** is a sequence of prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, an allocation $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$, and transfers $\hat{t}_1, \hat{t}_2$ such that

- Given $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, consumer i, $i = 1, 2$, chooses $\hat{c}_0^i, \hat{c}_1^i, \hat{c}_2^i, \ldots$ to solve

$$\max \ \sum_{t=0}^{\infty} \beta_i^t \log c_t^i$$
$$\text{s.t. } \sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t w_t^i + \hat{t}_i$$
$$c_t^i \geq 0.$$

- $\hat{c}_t^1 + \hat{c}_t^2 \leq w_t^1 + w_t^2$, $=$ if $\hat{p}_t > 0$, $t = 0, 1, 2, \ldots$.

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Characterization of equilibrium with transfers using calculus:

Once again, we use the Kuhn-Tucker theorem to show that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;$ $\hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{t}_1, \hat{t}_2$ is an equilibrium with transfers if and only if there exist Lagrange multipliers $\hat{\lambda}^1, \hat{\lambda}^2$, $\hat{\lambda}^i > 0$, such that

- $\beta_i^t \dfrac{1}{\hat{c}_t^i} = \hat{\lambda}_i \hat{p}_t$, $i = 1, 2$, $t = 0, 1, 2, \ldots$

- $\sum_{t=0}^{\infty} \hat{p}_t c_t^i = \sum_{t=0}^{\infty} \hat{p}_t w_t^i + \hat{t}_i$, $i = 1, 2$

- $\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$, $t = 0, 1, 2, \ldots$.

Second welfare theorem:

Suppose that $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is a Pareto efficient allocation where each consumer receives strictly positive consumption. Then there exist prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$ and transfers $\hat{t}_1, \hat{t}_2$ such that $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{t}_1, \hat{t}_2$ is an equilibrium.

Proof:

Since $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$ is a Pareto efficient allocation equilibrium, we know that there exist welfare weights $\hat{\alpha}_1, \hat{\alpha}_2$, $\hat{\alpha}_i \geq 0$, and Lagrange multipliers $\hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, \ldots$, $\hat{\pi}_t > 0$, such that

$$\hat{\alpha}_i \beta_i^t \frac{1}{\hat{c}_t^i} = \hat{\pi}_t$$

$$\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2 .$$

Since $\hat{c}_t^i > 0$, we know that $\hat{\alpha}_i > 0$, $i = 1, 2$. We also know that, if there exist prices $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots$, transfers $\hat{t}_1, \hat{t}_2$, and Lagrange multipliers $\hat{\lambda}_1, \hat{\lambda}_2$, $\hat{\lambda}_i > 0$, such that

$$\beta_i^t \frac{1}{\hat{c}_t^i} = \hat{\lambda}_i \hat{p}_t$$

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i = \sum_{t=0}^{\infty} \hat{p}_t w_t^i + \hat{t}_i$$

$$\hat{c}_t^1 + \hat{c}_t^2 = w_t^1 + w_t^2$$

then $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{t}_1, \hat{t}_2$ is an equilibrium with transfers. (In other words, we are given $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots;\ \hat{\alpha}_1, \hat{\alpha}_2$; and $\hat{\pi}_0, \hat{\pi}_1, \hat{\pi}_2, \ldots$ that satisfy certain properties, and we want to construct $\hat{p}_0, \hat{p}_1, \hat{p}_2, \ldots;\ \hat{t}_1, \hat{t}_2$; and $\hat{\lambda}_1, \hat{\lambda}_2$ that, together with $\hat{c}_0^1, \hat{c}_1^1, \hat{c}_2^1, \ldots;\ \hat{c}_0^2, \hat{c}_1^2, \hat{c}_2^2, \ldots$, satisfy certain other properties.) To prove the theorem, we set

$$\hat{p}_t = \hat{\pi}_t$$

$$\hat{\lambda}_i = \frac{1}{\hat{\alpha}_i}$$

$$\hat{t}_i = \sum_{t=0}^{\infty} \hat{p}_t c_t^i - \sum_{t=0}^{\infty} \hat{p}_t w_t^i . \qquad \blacksquare$$