Economics 8107
Macroeconomic Theory
Recitation 4

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February 13, 2017

Adapted From Manuel Amador's Notes and Aiyagari (1993)

Consider the following Bellman's equation of the income fluctuations problem:

$$v(x) = \max_{a \geq -\phi} \left\{ u(x-a) + \beta \sum_s \pi(s) v(Ra + y(s)) \right\}$$

where u is continuous, strictly increasing, strictly concave, and differentiable.

Define

$$\begin{aligned} \hat{a} &= a + \phi \\ z &= x + \phi \end{aligned}$$

We also have to make a transformation for $y(s)$. Note:

$$\begin{aligned} z' &= Ra + y(s) + \phi \\ &= R(\hat{a} - \phi) + y(s) + \phi \\ &= R\hat{a} + y(s) - (R-1)\phi \\ &= R\hat{a} + \tilde{y}(s) \end{aligned}$$

Thus, we define $\tilde{y}(s) = y(s) - r\phi$. So $c = x - a = z - \hat{a}$. The Bellman's equation becomes

$$v(z) = \max_{\hat{a} \geq 0} \left\{ u(z - \hat{a}) + \beta \sum_s \pi(s) v(R\hat{a} + \tilde{y}(s)) \right\}$$

Claim 0.1. *Let $z_{min} \equiv y_{min} - r\phi$. Then $c_t > 0$ whenever $z_t > z_{min}$.*

Claim 0.2. *The value function, v, is continuous, strictly increasing, strictly concave, and differentiable.*

If λ is the lagrange multiplier on the borrowing constraint, then the first order condition is

$$u'(c(z)) = \beta R \mathbb{E} v'\left(R\hat{a}(z) + \tilde{y}(s)\right) + \lambda$$
$$u'(c(z)) \geq \beta R \mathbb{E} v'\left(R\hat{a}(z) + \tilde{y}(s)\right)$$

Envelope:

$$v'(z) = u'(c(z))$$

Claim 0.3. *Consumption is strictly increasing in cash-in-hand, i.e.* $\frac{\partial c(z)}{\partial z} > 0$.

Proof. Consider the Envelope Condition:

$$\begin{aligned} v'(z) &= u'\left(c(z)\right) \\ v''(z) &= u''\left(c(z)\right)\frac{\partial c(z)}{\partial z} \end{aligned}$$

and so

$$\frac{\partial c(z)}{\partial z} = \frac{v''(z)}{u''\left(c(z)\right)} > 0$$

since u and v are strictly concave. □

Claim 0.4. *Assume that either* $U'(0) < \infty$ *or* $z_{min} = y_{min} - r\phi > 0$. *Then there is a* $\hat{z} > z_{min}$ *such that for all* $z_t \leq \hat{z}$, $c_t = z_t$ *and* $\hat{a}_{t+1} = 0$.

Proof. Either of the antecedents give us that $u'(z_{min})$ is finite, which implies that $v'(z_{min})$ is finite. Suppose for a contradiction that the borrowing constraint never binds. Then, we can combine the first order condition and the envelope condition to get, for some $z > z_{min}$.

$$\begin{aligned} v'(z) &= u'(c(z)) \\ &= \beta R \mathbb{E} v'\left(R\hat{a}(z) + \tilde{y}(s)\right) \\ &\leq \beta R v'\left(R\hat{a}(z) + \tilde{y}_{min}\right) \\ &< v'\left(z_{min}\right) \end{aligned}$$

If we take the limit of this inequality as $z \to z_{min}$, we get a contradiction. Thus, there must be some $\hat{z} > z_{min}$ where the borrowing constraint binds. This implies that $\hat{a}(\hat{z}) = 0$. Now, take any value for cash-in-hand, $z \leq \hat{z}$. We want to show that if $z < \hat{z}$, then $\hat{a}(z) = 0$. Suppose, by contradiction, that $\hat{a}(z) > \hat{a}(\hat{z})$. From the FOCs:

$$\begin{aligned} u'(c(z)) &= \beta R \mathbb{E} v'\left(R\hat{a}(z) + \tilde{y}(s)\right) \\ u'(c(\hat{z})) &\geq \beta R \mathbb{E} v'\left(R\hat{a}(\hat{z}) + \tilde{y}(s)\right) \end{aligned}$$

But as v' is strictly decreasing (v is strictly concave), we have

$$\begin{aligned} \beta R \mathbb{E} v'\left(R\hat{a}(z)+\tilde{y}(s)\right) &< \beta R \mathbb{E} v'\left(R\hat{a}(\hat{z})+\tilde{y}(s)\right) \\ u'(c(z)) &< u'(c(\hat{z})) \\ c(z) &> c(\hat{z}) \end{aligned}$$

since u' is strictly decreasing (u is strictly concave).

Since $c(\cdot)$ is strictly increasing, $z > \hat{z}$, which is a contradiction.

Thus, $\hat{a}(z) = 0, \forall z \leq \hat{z}$. □

Claim 0.5. *For all $z > \hat{z}$, $\frac{\partial \hat{a}(z)}{\partial z} > 0$, and both $\frac{\partial c(z)}{\partial z} \leq 1$ and $\frac{\partial \hat{a}(z)}{\partial z} \leq 1$.*

Proof. For $z > \hat{z}$, the borrowing constraint is not binding. The first-order condition is

$$\begin{aligned} u'(c(z)) &= \beta R \mathbb{E} v'\left(R\hat{a}(z)+\tilde{y}(s)\right) \\ u''\left(c(z)\right)\frac{\partial c(z)}{\partial z} &= \beta R^2 \mathbb{E} v''\left(R\hat{a}(z)+\tilde{y}(s)\right)\frac{\partial \hat{a}(z)}{\partial z} \end{aligned}$$

and so

$$\frac{\partial \hat{a}(z)}{\partial z} = \frac{u''\left(c(z)\right)\frac{\partial c(z)}{\partial z}}{\beta R^2 \mathbb{E} v''\left(R\hat{a}(z)+\tilde{y}(s)\right)} > 0$$

Finally,

$$\begin{aligned} c(z)+\hat{a}(z) &= z \\ \frac{\partial c(z)}{\partial z}+\frac{\partial \hat{a}(z)}{\partial z} &= 1 \end{aligned}$$

Since both functions are strictly increasing, $\frac{\partial c(z)}{\partial z} \leq 1$ and $\frac{\partial \hat{a}(z)}{\partial z} \leq 1$. □

Claim 0.6. *If (i) $\beta R < 1$, (ii) $y(s)$ has bounded support, and (iii) $-\frac{cu''(c)}{u'(c)}$ is bounded above for all sufficiently large c, then there exists a z^* such that for all $z_t \geq z^*$, $z_{t+1} \leq z_t$.*

We want to show that there exits $z^* > \hat{z}$ such that

$$\forall z \geq z^*, z'_{\max}(z) \equiv R\hat{a}(z)+\tilde{y}_{\max} \leq z$$

where $z'_{\max}(z)$ is the maximum cash-in-hand tomorrow given z today.

For $z > \hat{z}$, the borrowing constraint is not binding. So the Euler's equation is

$$\begin{aligned} u'\left(c(z)\right) &= \beta R \mathbb{E} u'(c(z'(z))) \\ u'\left(c(z)\right) &= \beta R \frac{\mathbb{E} u'(c(z'(z)))}{u'\left(c\left(z'_{\max}(z)\right)\right)} u'\left(c\left(z'_{\max}(z)\right)\right) \end{aligned}$$

Suppose that

$$\lim_{z\to\infty} \frac{\mathbb{E}u'(c(z'(z)))}{u'\left(c\left(z'_{\max}(z)\right)\right)} = 1$$

So for a sufficiently large $z^* > \hat{z}$, $\forall z \geq z^*$, $\frac{\mathbb{E}u'(c(z'(z)))}{u'(c(z'_{\max}(z)))} \approx 1$. Given that $\beta R < 1$,

$$\begin{aligned} u'(c(z)) &\leq u'\left(c\left(z'_{\max}(z)\right)\right) \\ c(z) &\geq c\left(z'_{\max}(z)\right) \end{aligned}$$

Since $c(\cdot)$ is increasing in z,

$$z \geq z'_{\max}(z)$$

So we need the condition that

$$\lim_{z\to\infty} \frac{\mathbb{E}u'(c(z'(z)))}{u'\left(c\left(z'_{\max}(z)\right)\right)} = 1$$

Since $z'_{\max}(z) \geq z'(z) \geq z'_{\min}(z)$,

$$1 \leq \frac{\mathbb{E}u'(c(z'(z)))}{u'\left(c\left(z'_{\max}(z)\right)\right)} \leq \frac{u'\left(c\left(z'_{\min}(z)\right)\right)}{u'\left(c\left(z'_{\max}(z)\right)\right)}$$

Recall that $\hat{a}\left(z'_{\max}(z)\right) \geq \hat{a}\left(z'_{min}(z)\right)$. So

$$\begin{aligned} z'_{\max}(z) - c\left(z'_{\max}(z)\right) &\geq z'_{\min}(z) - c\left(z'_{\min}(z)\right) \\ c\left(z'_{\min}(z)\right) &\geq z'_{\min}(z) - z'_{\max}(z) + c\left(z'_{\max}(z)\right) \\ c\left(z'_{\min}(z)\right) &\geq \tilde{y}_{\min} - \tilde{y}_{\max} + c\left(z'_{\max}(z)\right) \\ c\left(z'_{\min}(z)\right) &\geq -\Delta + c\left(z'_{\max}(z)\right) \end{aligned}$$

So

$$1 \leq \frac{\mathbb{E}u'(c(z'(z)))}{u'\left(c\left(z'_{\max}(z)\right)\right)} \leq \frac{u'\left(c\left(z'_{\max}(z)\right) - \Delta\right)}{u'\left(c\left(z'_{\max}(z)\right)\right)}$$

Thus, we need a utility function where

$$\lim_{c\to\infty} \frac{u'(c-A)}{u'(c)} = 1$$

Power (CRRA) will do: $u = \frac{c^{1-\sigma}-1}{1-\sigma}$ since

$$\frac{(c-A)^{-\sigma}}{c^{-\sigma}} = \left[1 - \frac{A}{c}\right]^{-\sigma} \to 1 \text{ as } c \to \infty$$

Claim 0.7. *Under the conditions we have given so far, there exists a unique invariant distribution and it is stable.*

Proof. Theorem 12.12 of SLP states: If a transition function P is monotone, has the Feller property and satisfies a "mixing condition," then there is a unique stable invariant distribution.

The relative markov process P, in this context, is given by

$$z_{t+1} = R\hat{a}(z_t) + y(s) - r\phi$$

One statement of the transition function being monotone is that for two probability measures λ, μ where μ first order stochastic dominates λ, then $T^*\mu$ dominates $T^*\lambda$. This is simply stating that if a higher value of z if more likely in time t, than it will still be more likely in time $t+1$. This is guaranteed in our environment since $\hat{a}$ is increasing in z. Another way to see this is to consider any increasing function g,

$$E[g(z_{t+1})|z_t] = E[g(R\hat{a}(z_t) + y(s) - r\phi)|z_t]$$

which demonstrates that $E[g(z_{t+1})|z_t]$ is an increasing function of z_t.

The Feller Property states that any continuous function integrated across the transition function must remain continuous. This implies that $E[g(z_{t+1})|z_t]$ must be continuous in z_t, which is a result of the continuity of $\hat{a}$.

The mixing condition is that there exists some $z \in [z_{min}, z_{max}]$, $\epsilon > 0$, and $T \geq 1$ such that $Prob\{z_T \in [z, z_{max}]|z_0 = z_{min}\} \geq \epsilon$ and $Prob\{z_T \in [z_{min}, z]|z_0 = z_{max}\} \geq \epsilon$. This property results from the observation that the unique fixed point of cash-in-hand as a function of y_{min} is z_{min} and that any fixed point for y_{max} must exceed z_{min}.

For the first point, note that $\hat{z} > z_{min}$, which we already proved. This implies that for any $z \leq \hat{z}$

$$\begin{aligned} z'(z|y_{min}) &= R0 + y_{min} - r\phi \\ &= z_{min} \end{aligned}$$

This proves there is a fixed point at z_{min}. And since, $z'(\hat{z}|y_{min}) = z_{min}$ and $\frac{\partial \hat{a}(z)}{\partial z} \leq 1$, there cannot be another fixed point.

Consider any fixed point,z, for y_{max}.

$$\begin{aligned} z &= R\hat{a}(z) + y_{max} - r\phi \\ &= R\hat{a}(z) + y_{max} - y_{min} + z_{min} \\ &> z_{min} \end{aligned}$$

The mixing condition is then simply established by showing that there exists a sequence of shocks that can take an agent from the z_{min} to the lowest fixed point for y_{max} and vice-versa. There is a more rigorous and general proof in Brock and Mirman (1972), but the technical machinery isn't necessary here.

□