Economics 8106
Macroeconomic Theory
Recitation 1

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Outline:

- Measure Theory
- Markov Processes
- Summary SLP Chapter 9

1 Some Math Premilinaries

Many of the models that we talk about for the rest of the year will include some feature of random variation. We would like to be able to use the dynamic programming results that we have already applied to deterministic settings to solve these more complex problems. For the most part, these results go through with nearly identical conditions. In this section, I will introduce some basic concepts of measure theory and Markov processes. The technical details are not terribly important, but it is good to be familiar with the vocabulary. You can find in-depth discussion of these subjects in Chapters 7 and 8 of SLP.

1.1 Measure Spaces and Measure Functions

So far, we have developed the notion of a metric and a metric space. A metric tells a distance between two general objects, e.g. functions, operations, etc. As you may guess, a measure captures the notion of length, area, volume, etc. The first step in "measuring" anything is to determine what we are able to know.

Definition 1.1. Let X be a set and $\mathscr{X} \subseteq 2^X$ be a family of subsets of X. $\mathscr{X}$ is called a **σ-algebra** if

1. $\emptyset, X \in \mathscr{X}$
2. $E \in \mathscr{X} \Rightarrow E^C = X \backslash E \in \mathscr{X}$ ($\mathscr{X}$ is closed under complement.)
3. $\forall n, E_n \in \mathscr{X} \Rightarrow \cup_{n=1}^{\infty} E_n \in \mathscr{X}$ ($\mathscr{X}$ is closed under countable union.)

A σ-algebra imposes certain consistency to the family of sets under consideration. Only subsets of the σ-algebra can be known, hence measured. First, it must be possible to know when none or all of the outcomes occurred. Also if there is an outcome that occurred it must be possible to determine if it didn't. Finally if it is possible to determine that some outcomes occurred individually it can also be determined if at least one or all of them were realized.

Definition 1.2. For any set X and σ-algebra $\mathscr{X}$, the pair $(X, \mathscr{X})$ is called a *measurable space*. Any set $E \in \mathscr{X}$ is *measurable*.

Definition 1.3. For any metric space (X, ρ), the **Borel algebra** is the smallest σ-algebra containing the open balls, i.e. containing all sets of the form $E = \{x \in X : \rho(x, x_0) < \delta\}$where $x_0 \in X$ and $\delta > 0$.

Definition 1.4. A ***measure*** is a function, $\mu : \mathscr{X} \to \mathbb{R}_+$, that satisfies the following conditions:

1. null set has measure zero; $\mu(\emptyset) = 0$,
2. $\mu(E) \geq 0, \forall E \in \mathscr{X}$
3. $\mu(\cdot)$ is *countably additive*;
$$\mu\left(\bigcup_{i \in I} E_i\right) = \sum_{i \in I} \mu(E_i),$$
for every disjoint countable collection of sets, I, in $\mathscr{X}$.

Definition 1.5. For any set X, σ-algebra $\mathscr{X}$, and measure μ, the triplet $(X, \mathscr{X}, \mu)$ is called a *measure space*. If $\mu(X) = 1$, then $\mu(\cdot)$ is called a *probability measure*. In this case, we call the triple $(X, \mathscr{X}, \mu)$ a *probability space*.

One can think of a function as mapping certain events in a given measure space to outcomes in another measure space. A function is measurable if the sets that induce a given outcome are measurable.

Definition 1.6. Given a measurable space $(X, \mathscr{X})$, a real-valued function $f : X \to \mathbb{R}$ is measurable w.r.t. $\mathscr{X}$ ($\mathscr{X}$-measurable) if

$$\{x \in X | f(x) \leq a\} \subseteq \mathscr{X}, \forall a \in \mathbb{R}$$

Definition 1.7. Given measurable spaces $(X, \mathscr{X})$ and $(Y, \mathscr{Y})$. Let $\Gamma : X \rightrightarrows Y$ be a correspondence. Then the function $h : X \to Y$ is a measurable selection from Γ if h is measurable and $h(x) \in \Gamma(x), \forall x \in X$.

Definition 1.8. Let $(X, \mathscr{X}, \mu)$ be a measure space. A proposition is said to hold *almost everywhere* (a.e.) or *almost surely* (a.s.) if there exists a set $E \in \mathscr{X}$ such that $\mu(E) = 0$ and the proposition holds in E^c.

1.2 Markov Processes

Nearly every stochastic process that we examine in economics can be characterized as a Markov Process. (i.i.d. is a special case of a Markov process). In words, a markov process is a stochastic process in which the probability of some future event depends only on the current event, rather than the entire history of the process. This is a convenient characteristic to include in models for obvious reasons. The key feature of a Markov process is a transition matrix, or more generally a transition function.

Definition 1.9. Let $(X, \mathscr{X})$ be a measurable space. A transition function is a function $Q : X \times \mathscr{X} \to [0, 1]$, such that:

1. for each $x \in X$, $Q(x, \cdot)$ is a probability measure on $(X, \mathscr{X})$,
2. and, for each $E \in \mathscr{X}$, $Q(\cdot, E)$ is a $\mathscr{X}$-measurable function.

Definition 1.10. A stationary stochastic process $\{x_t\}$ is a *markov process* if for $n \geq 1$,

$$Pr(x_{t+n}|x_t, x_{t-1}, \ldots, x_0) = Pr(x_{t+n}|x_t)$$

For a markov process with transition function Q,

$$Pr(x_{t+1} \in X|x_t) = Q(x_t, X)$$

In a discrete setting, the transition function is often characterized by a transition matrix P where an element of the matrix p_{ij} is such that,

$$Pr(x_{t+1} = j|x_t = i) = p_{ij}$$

Just as in the deterministic case, continuity is important for proving properties of the value function. We need to define a kind of continuity property for the stochastic process in the model.

Definition 1.11. A transition function Q on $(Z, \mathscr{X})$ has the *Feller Property* if the associated operator T maps the space of bounded functions on Z into itself, i.e. $T : C(Z) \to C(Z)$. The associated operator T is defined as

$$Tf(z) = \int f(z')Q(z, dz'), \text{ all } z \in Z$$

2 Stochastic Dynamic Programming

In this section, we will take an abbreviated look at Chapter 9 of SLP. It is important that you be able to read this chapter and implement its findings. My goal here is to take the edge off so that you feel comfortable investigating further on your own.

2.1 9.1 Principle of Optimality

First let's introduce the spaces we are going to be working with.

$$\begin{aligned}(X, \mathscr{X}) &\text{ - A measurable space for the endogenous state}\\ (Z, \mathscr{Z}) &\text{ - A measurable space for the exogeneous shock}\\ (S, \mathscr{S}) &= (X \times Z, \mathscr{X} \times \mathscr{Z})\end{aligned}$$

We will assume that the exogenous state follows a stationary Markov process, with transition function $Q(\cdot)$, defined on $(Z, \mathscr{Z})$.

Given today's state, s, we will denote the set of feasible next period states by $\Gamma(s)$; $\Gamma : S \to X$ is the feasibility correspondence. Let A be the graph of $\Gamma(\cdot)$:

$$\begin{aligned}A &= \{(x, y, z) \in X \times X \times Z \mid y \in \Gamma(x, z)\}\\ \mathscr{A} &= \{C \in \mathscr{X} \times \mathscr{X} \times \mathscr{Z} \mid C \subseteq A\}\end{aligned}$$

In selection from A, x is today's state, y is tomorrows state, and z is the exogenous shock.

We will define $F : A \to \mathbb{R}$ is one-period payoff function (think utility function), and $\beta \geq 0$ is the discount factor. You can imagine the function F being something like

$$F(x, y, z) = u(zf(x) - y)$$

We are going to be focusing on problems that can be written in the form

$$v(s) = v(x, z) = \sup_{y \in \Gamma(s)} \left\{ F(x, y, z) + \beta \int_Z v(y, z') Q(z, dz') \right\}. \tag{1}$$

Definition 2.1. A plan is a value $\pi_0 \in X$ and a sequence of measurable functions $\pi_t : Z^t \to X$, for $t = 1, 2, \ldots$.
Definition 2.2. A plan π is feasible from $s_0 \in S$, if:

1. $\pi_0 \in \Gamma(s_0)$,
2. and, $\pi_t(z^t) \in \Gamma[\pi_{t-1}(z^{t-1}), z_t]$, for all $z^t \in Z^t$ and $t = 1, 2, \ldots$.

We will denote the set of all feasible plans from s_0 by $\Pi(s_0)$. Recall from the deterministic case that, our first requirement for the sequence problem to be well defined was for $\Gamma(\cdot)$ to be a non-empty correspondence. Here, we need a stronger assumption that ensures the existence of *measurable selections*. This is done in the following assumptions.

Assumption. ***9.1***

$\Gamma(\cdot)$ is non-empty valued, and A is $(\mathscr{X} \times \mathscr{X} \times \mathscr{Z})$-measurable. Moreover, $\Gamma(\cdot)$ has a measurable selection; i.e. there exists a measurable function $h : S \to X$ such that $h(s) \in \Gamma(s)$ for all $s \in S$.

Next, we need an analogous assumption to 4.2, which says that the period 0 utility for the sequential problem is well defined for any feasible plan (although it can be plus or minus infinity). To do so, we have to guarantee that the planner can calculate the expectation

Assumption. ***9.2***

$F : A \to \mathbb{R}$ is $\mathscr{A}$-measurable, and one of the followings holds:

1. *$F \geq 0$ or $F \leq 0$.*
2. *For each $s_0 = (x_0, z_0) \in S$ and each plan $\pi \in \Pi(s_0)$, $F[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t]$ is $\mu^t(z_0, \cdot)$-integrable, for $t = 1, 2, \ldots$, and the limit*

$$F(x_0, \pi_0, z_0) + \lim_{n\to\infty} \sum_{t=1}^{n} \int_{Z^t} \beta^t F\left[\pi_{t-1}\left(z^{t-1}\right), \pi_t\left(z^t\right), z_t\right] \mu^t\left(z_0, dz^t\right)$$

exists (although it might not be bounded).

Notice that, under Assumptions 9.1 and 9.2, discounted expected payoff (in a planner's problem) is well-defined, and we can define the sequential planner's problem as:

$$v^*(s) = \sup_{\pi\in\Pi(s)} \left\{ F(x_0, \pi_0, z_0) + \sum_{t=1}^{\infty} \int_{Z^t} \beta^t F\left[\pi_{t-1}\left(z^{t-1}\right), \pi_t\left(z^t\right), z_t\right] \mu^t\left(z_0, dz^t\right) \right\}. \quad (2)$$

Definition 2.3. If there exists a function $v(\cdot)$ that solves this functional equation, then, we can define the associated policy correspondence as:

$$G(x, z) = \left\{ y \in \Gamma(x, z) \mid v(x, z) = F(x, y, z) + \beta \int_Z v(y, z') Q(z, dz') \right\}. \quad (3)$$

The meat of this section is to show that the functions defined in (1) and (2) are equivalent and that there solutions are equivalent. In Chapter 4, Theorem 4.2 showed that the supremum function v^* satisfies the function equation for v and 4.4 that the sequential solution was a solution to the functional equation all along the path. There is no analogous version of Theorem 4.2 in this setting because of measurability issues, and the stochastic analogy for theorem 4.4 requires a few more additional assumptions/definitions. We do however have relatively neat analogies for Theorem 4.3 and 4.5 which show the converse.

Theorem. ***9.2 (Principle of Optimality–I)***

Suppose $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, Q, Γ, F, and β satisfy Assumptions 9.1 and 9.2. Let v^* be the solution to 2, and $v(\cdot)$ be a measurable function that solves 1, so that:

$$\lim_{t\to\infty}\int_{Z^t}\beta^t v\left[\pi_{t-1}\left(z^{t-1}\right), z_t\right]\mu^t\left(z_0, dz^t\right) = 0,$$

for all $\pi \in \Pi(s_0)$, and all $s_0 \in S$. Let $G(\cdot)$ be the correspondence defined by 3, which is non-empty and permits a measurable selection. Then $v^* = v$, and the plan generated by $G(\cdot)$ attains the supremum in 2.

For the partial converse to Theorem 9.2, we need to strengthen Assumption 9.2.
Assumption. ***9.3*** *If F takes on both signs, there is a collection of nonnegative, measurable functions $L_t : S \to \mathbb{R}_+$, $t = 0, 1, \ldots$, such that for all $\pi \in \Pi(s_0)$ and all $s_0 \in S$*

$$|F(x_0, \pi_0, z_0)| \leq F_0(s_0);$$
$$|F[pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t]| \leq L_t(s_0), \text{ all } z_t \in Z^t,\ t = 1, 2, \ldots$$

and

$$\sum_{t=0}^{\infty}\beta^t L_t(s_0) < \infty$$

Theorem. ***9.4 (Principle of Optimality–II)***

Suppose $(X, \mathscr{X})$, $(Z, \mathscr{Z})$, Q, Γ, F, and β satisfy Assumptions 9.1 through 9.3. Let $v^*(\cdot)$ be the solution to (2). Assume that v^* is measurable and satisfies (1), and define G by (3). Assume that G is nonempty and permits a measurable selection. Let $(x_0, z_0) = s_0 \in S$, and let $\pi^* \in \Pi(s_0)$ be a plan that attains the supremum in (2) for initial condition s_0. Then there exists a plan π^G generated by G from s_0 such that

$$\pi_0^G = \pi_0^*, \text{ and}$$
$$\pi_t^G(s^t) = \pi_t^*(z^t),\ \mu^t(z_0, \cdot)\text{-a.e.},\ t = 1, 2, \ldots$$

2.2 Bounded Returns

Next step, is to ensure the existence of a solution to the functional equation. Like the deterministic case, when the return function is bounded, there is a good chance that this is the case. In this section, we consider the fairly general assumptions under which, this is the case, by focusing on the case of bounded returns.
Assumption. ***9.4*** *(analog of A 4.3)*

X is a convex Borel set in $\mathbb{R}^l$, and $\mathscr{X}$ is its Borel subsets.

Assumption. ***9.5***

One of the followings holds:

1. *Z is a countable set, and $\mathscr{Z}$ is the σ-algebra containing all of its subsets.*
2. *Z is a compact Borel set in $\mathbb{R}^k$, with its Borel subsets $\mathscr{Z}$, and the transition function $Q(\cdot)$ has the Feller property (SLP Chapter 8.1).*

The key role of Assumption 9.5 is to ensure that the integral in 1,

$$Mf(y,z) = \int_Z v(y,z') Q(z,dz'), \text{ for all } (y,z) \in X \times Z, \tag{4}$$

maps a bounded continuous function $v : X \times Z \to \mathbb{R}$ into the space of bounded continuous functions over $X \times Z$. Moreover, by Lemma 9.5, assumptions 9.4 and 9.5 ensure that, if $v(\cdot)$ is increasing or concave, then the integral would be an increasing or concave function of (y,z).

Given this property, the rest is quite similar to the stochastic case; first, we may use Blackwell's sufficient conditions to ensure that the mapping defined by 1 is a contraction, and then use the Contraction Mapping Theorem to ensure the existence of a fixed point. First we need the following two assumptions:
Assumption. ***9.6*** *(analog of A 4.3)*

The correspondence $\Gamma : X \times Z \to X$ is non-empty, compact-valued, and continuous.
Assumption. ***9.7****(analog of A 4.4)*

The function $F : A \to \mathbb{R}$ is bounded and continuous, and $\beta \in (0,1)$.

Now, we have the following theorem:
Theorem. ***9.6 Existence & Uniqueness***

Under Assumptions 9.4-9.7, the operator T, defined by

$$Tf(x,z) = \sup_{y \in \Gamma(s)} \left\{ F(x,y,z) + \beta \int_Z f(y,z') Q(z,dz') \right\}, \tag{5}$$

maps the set of bounded continuous functions, $C(S)$, into itself, and has a unique fixed point in this set, $v(\cdot) \in C(S)$. Moreover, for all $v_0(\cdot) \in C(S)$:

$$\|T^n v_0 - v\| \leq \beta^n \|v_0 - v\|, \quad n = 1, 2, \ldots.$$

In addition, the correspondence $G(\cdot)$ defined by 3 is non-empty, compact-valued, and upper hemi-continuous.

2.3 Inheriting Properties of the Value Function

If the operator M in 4 preserves the monotonicity and concavity of the integrand, it is natural to expect that the value function inherits these properties from the payoff function; what we had in the deterministic case, as well. To formalize this idea, let us introduce the following assumptions. Note that, A_i denotes the i-section of the set A, in what follows.

Assumption. ***9.8***

For each $(y,z) \in X \times Z$, $F(\cdot, y, z) : A_{yz} \to \mathbb{R}$ is strictly increasing.

Assumption. ***9.9***

For each $z \in Z$, $x \leq x'$ implies $\Gamma(x,z) \in \Gamma(x',z)$.

Now, we have our first inheritance property of the value function:

Theorem. ***9.7*** *(analog of Thm 4.7)*

Under Assumptions 9.4-9.9, the fixed point of operator T in 5 is strictly increasing in x, for each $z \in Z$.

The value function inherits the concavity of the payoff function as well:

Assumption. ***9.10***

For each $z \in Z$, $F(\cdot, \cdot, z) : A_z \to \mathbb{R}$ is strictly concave in (x,y).

Assumption. ***9.11***

The set A_z is convex.

Theorem. ***9.8*** *(analog of Thm 4.8)*

Under Assumptions 9.4-9.7 and 9.10-9.11, the fixed point of operator T in 5 is strictly concave in x, for each $z \in Z$, and the corresponding policy correspondence is a continuous function.

Finally, $v(\cdot)$ inherits the differentiability of the payoff function, too:

Assumption. ***9.12***

For a fixed $z \in Z$, $F(\cdot, \cdot, z)$ is continuously differentiable in (x,y), in the interior of A_z.

Theorem. ***9.10*** *(analog of Thm 4.11)*

Suppose Assumptions 9.4-9.7 and 9.10-9.12 hold, $v(\cdot) \in C(S)$ is the fixed point of operator T in 5, and $g : S \to X$ is the corresponding value function. If $x_0 \in intX$, and $g(x_0, z_0) \in int\Gamma(x_0, z_0)$, then $v(\cdot, z_0)$ is continuously differentiable in x at x_0, with derivatives given by

$$v_i(x_0, z_0) = F_i[x_0, g(x_0, z_0), z_0], \text{ for } i = 1, 2, \ldots, l.$$