Economics 8105
Macroeconomic Theory
Recitation 5

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October 4th, 2016

Outline:

- Defining a TDCE
- Characterizing a TDCE Equilibrium
- Solving an Equivalent Planner Problem

1 Taxation in a Production Economy

1.1 Defining Equilibrium

Definition 1.1. An **Tax Distorted Competitive Equilibrium** is

- an allocation for the HH: $z^H = \{(c_t, l_t, n_t, k_t, x_t)\}_{t=0}^{\infty}$
- an allocation for the firm: $z^F = \{(y_t^f, k_t^f, n_t^f)\}_{t=0}^{\infty}$
- a system of prices: $p = \{(p_t, w_t, r_t)\}_{t=0}^{\infty}$
- a government policy: $g = \{(g_t, \tau_{ct}, \tau_{xt}, \tau_{kt}, \tau_{nt}, T_t)\}_{t=0}^{\infty}$

such that

(HH) Given p and g, z^H solves

$$\max_{c_t,l_t,n_t,k_t,x_t} \quad \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

$$s.t.$$

$$\sum_{t=0}^{\infty} p_t(1+\tau_{ct})c_t + p_t(1+\tau_{xt})x_t \leq \sum_{t=0}^{\infty} w_t(1-\tau_{nt})n_t + r_t(1-\tau_{kt})k_t + T_t$$

$$k_{t+1} \leq x_t + (1-\delta)k_t, \ \forall t$$

$$l_t + n_t \leq 1, \ \forall t$$

$$c_t, k_{t+1}, l_t, n_t \geq 0, \ \forall t$$

$$k_0 > 0, \quad \text{given}$$

(Firm) Given p, z^F solves

$$\max_{\{(y_t^f, k_t^f, n_t^f)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left[p_t y_t^f - w_t n_t^f - r_t k_t^f \right]$$

$$s.t.$$

$$y_t^f \leq F(k_t^f, n_t^f), \ \forall t$$

$$k_t^f, n_t^f, y_t^f \geq 0, \forall t$$

(Mkt) For all t,

$$\text{(Goods Market)} \quad c_t + x_t + g_t = y_t^f \leq F(k_t^f, n_t^f)$$

$$\text{(Labor Market)} \quad n_t = n_t^f$$

$$\text{(Capital Market)} \quad k_t = k_t^f$$

(Govt)

$$\sum_{t=0}^{\infty} p_t g_t + T_t = \sum_{t=0}^{\infty} p_t \tau_{ct} c_t + p_t \tau_{xt} x_t + w_t \tau_{nt} n_t + r_t \tau_{kt} k_t$$

1.2 First Order Conditions

We will assume the typical things about u and F so that the solution is interior, the appropriate constraints are binding, profits are zero, and we can make some convenient substitutions.

$$x_t = k_{t+1} - (1-\delta)k_t$$

$$l_t = 1 - n_t$$

In the following algebra, I will use a bit of abbreviated notation. Function subscripts will denote partial derivatives, i.e. $\partial F(k_t, n_t)/\partial k = F_k(k_t, n_t)$ and $\partial u(c_t, l_t)/\partial c = u_c(c_t, l_t)$. I will also use a shorthand for evaluating marginal utility in a given time period, $u_c(c_t, l_t) = u_c(t)$. The household problem becomes,

$$\max_{c_t, n_t, k_t} \quad \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

$$s.t.$$

$$\sum_{t=0}^{\infty} p_t(1+\tau_{ct})c_t + p_t(1+\tau_{xt})(k_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty} w_t(1-\tau_{nt})n_t + r_t(1-\tau_{kt})k_t + T_t$$

$$n_t \leq 1, \ \forall t$$

$$c_t, k_{t+1}, n_t \geq 0, \ \forall t$$

$$k_0 \ > 0, \ \text{ given}$$

The household first order conditions are

$$\beta^t u_c(t) = p_t(1+\tau_{ct})\lambda \qquad (c_t)$$

$$\beta^t u_l(t) = w_t(1-\tau_{nt})\lambda \qquad (n_t)$$

$$p_t(1+\tau_{xt})\lambda = p_{t+1}(1+\tau_{xt+1})(1-\delta)\lambda + r_{t+1}(1-\tau_{kt+1})\lambda \qquad (k_{t+1})$$

The firm first order conditions are

$$r_t = p_t F_k(t) \qquad (k_t^f)$$

$$w_t = p_t F_n(t) \qquad (n_t^f)$$

To derive to Euler Equation, we want to relate the marginal value of consumption today to the marginal value tomorrow. So we will divide the FOC for c_t in period t by the FOC in period $t+1$.

$$\frac{u_c(t)}{u_c(t+1)} = \beta \frac{p_t(1+\tau_{ct})}{p_{t+1}(1+\tau_{ct+1})} \qquad (1)$$

We can use the "no arbitrage" condition from the k_{t+1} FOC to relate prices between periods.

$$p_t(1+\tau_{xt}) = p_{t+1}(1+\tau_{xt+1})(1-\delta) + r_{t+1}(1-\tau_{kt+1})$$

$$\frac{p_t(1+\tau_{xt})}{p_{t+1}(1+\tau_{xt+1})} = 1 - \delta + \frac{r_{t+1}(1-\tau_{kt+1})}{p_{t+1}(1+\tau_{xt+1})}$$

$$\frac{p_t(1+\tau_{xt})}{p_{t+1}(1+\tau_{xt+1})} = 1 - \delta + \frac{(1-\tau_{kt+1})}{(1+\tau_{xt+1})} F_k(t+1)$$

$$\frac{p_t}{p_{t+1}} = \frac{(1+\tau_{xt+1})}{(1+\tau_{xt})}[1 - \delta + \frac{(1-\tau_{kt+1})}{(1+\tau_{xt+1})} F_k(t+1)]$$

We can now use this to substitute the prices out of (1) to get the Euler Equation in terms of only allocations.

$$\frac{u_c(t)}{u_c(t+1)} = \beta \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} \frac{(1+\tau_{xt+1})}{(1+\tau_{xt})}[1-\delta+\frac{(1-\tau_{kt+1})}{(1+\tau_{xt+1})}F_k(t+1)]$$

The intratemporal marginal substitution between consumption and labor can be found in a similar way.

$$\frac{u_l(t)}{u_c(t)} = \beta \frac{w_t(1-\tau_{nt})}{p_t(1+\tau_{ct})}$$

Substituting the firm's FOC for labor, we get

$$\frac{u_l(t)}{u_c(t)} = \beta \frac{(1-\tau_{nt})}{(1+\tau_{ct})}F_n(t)$$

1.3 Characterizing Equilibrium

Now we are close to fully characterizing the equilibrium for this economy. If u is strictly concave and we have a unique solution, then we have one equation characterizing the ratio between consumption and labor and one equation governing the evolution over time. We also have the resource constraint, the transversality condition, and the initial condition. All in all, we have

$$\frac{u_c(t)}{u_c(t+1)} = \beta \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} \frac{(1+\tau_{xt+1})}{(1+\tau_{xt})}[1-\delta+\frac{(1-\tau_{kt+1})}{(1+\tau_{xt+1})}F_k(t+1)] \tag{2}$$

$$\frac{u_l(t)}{u_c(t)} = \beta \frac{(1-\tau_{nt})}{(1+\tau_{ct})}F_n(t) \tag{3}$$

$$c_t + k_{t+1} + g_t = F(k_t, n_t) + (1-\delta)k_t \tag{4}$$

$$\lim_{T\to\infty} \beta^T u_c(T)k_{T+1} = 0 \tag{5}$$

$$k_0 \text{ given} \tag{6}$$

The Euler Equation (1) and the resource constraint (4) can be combined to create a second order difference equation for capital stock. And the consumption/labor substitution equation (3) can be used to identify labor in each period, given k_t and $k_t + 1$. Then the initial condition (6) and the transversality condition (5) give us the boundaries that we need to fully characterize the solution. So we are done!

2 Solving a TDCE with a Planner Problem

In certain circumstances, we can express a TDCE as a social planners problem. The key question is whether we can construct a planner's problem that will give us the same set of equations as in Section 1.3. We will consider one example here. Suppose $\tau_{xt} = \tau_{ct} = 0$, $\tau_{nt} = \tau_{kt} = \tau$, $T_t = 0$, and the government spends all of its revenue each period on government purchases, g_t. We can insert this policy into equations from Section 1.3 to characterize the TDCE solution.

$$\frac{u_c(t)}{u_c(t+1)} = \beta[1 - \delta + (1-\tau)F_k(t+1)] \tag{7}$$

$$\frac{u_l(t)}{u_c(t)} = (1-\tau)F_n(t) \tag{8}$$

$$c_t + k_{t+1} + g_t = F(k_t, n_t) + (1-\delta)k_t \tag{9}$$

$$\lim_{T\to\infty} \beta^T u_c(T) k_{T+1} = 0 \tag{10}$$

$$k_0 \text{ given} \tag{11}$$

I claim that, for some new production function, $\hat{F}$, the following social planner problem will obtain the same solution, and then we will demonstrate it.

$$\begin{aligned}
\max_{c_t, n_t, k_t} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t) \\
s.t. \quad & \\
c_t + k_{t+1} &\leq \hat{F}(k_t, n_t) + (1-\delta)k_t \ \forall t \\
n_t &\leq 1, \ \forall t \\
c_t, k_{t+1}, n_t &\geq 0, \ \forall t \\
k_0 &> 0, \quad \text{given}
\end{aligned}$$

2.1 Characterizing the Equilibrium

The first order conditions of this planner problem are

$$\beta^t u_c(t) = \mu_t \tag{c_t}$$

$$\beta^t u_l(t) = \hat{F}_n(t)\mu_t \tag{n_t}$$

$$\mu_t = \mu_{t+1}((1-\delta) + \hat{F}_k(t+1)) \tag{k_{t+1}}$$

You can think of the lagrange multiplier on the resource constraint as similar to the Arrow price. It reflects the value to marginal utility of relaxing the resource constraint, and thus being able to consume one more marginal unit. The analogous arbitrage condition is

$$\frac{\mu_t}{\mu_{t+1}} = ((1-\delta) + \hat{F}_k(t+1))$$

Using the same methodology we used in Section 1.2, we can find the equations that characterize the equilibrium to the planner problem.

$$\frac{u_c(t)}{u_c(t+1)} = \beta[1 - \delta + \hat{F}_k(t+1)] \tag{12}$$

$$\frac{u_l(t)}{u_c(t)} = \hat{F}_n(t) \tag{13}$$

$$c_t + k_{t+1} = \hat{F}(k_t, n_t) + (1-\delta)k_t \tag{14}$$

$$\lim_{T\to\infty} \beta^T u_c(T) k_{T+1} = 0 \tag{15}$$

$$k_0 \text{ given} \tag{16}$$

2.2 Demonstrating Equivalence

In order to show that these problems are equivalent, we need to show that equations (7) through (11) are equivalent to (12) through (16) and thereby identify the same allocations. It is obvious that (15) and (11) are identical to (15) and (16). In order to make the other equations fit, we have to find the appropriate $\hat{F}$. By eyeballing the other equations, it may appear that a good candidate is $\hat{F}(k_t, n_t) = (1-\tau)F(k_t, n_t)$. We can show that this will work in each equation.

With this definition, (12) and (13) become

$$\frac{u_c(t)}{u_c(t+1)} = \beta[1 - \delta + \hat{F}_k(t+1)]$$
$$\frac{u_c(t)}{u_c(t+1)} = \beta[1 - \delta + (1-\tau)F_k(t+1)]$$

$$\frac{u_l(t)}{u_c(t)} = \hat{F}_n(t)$$
$$\frac{u_l(t)}{u_c(t)} = (1-\tau)F_n(t)$$

So far so good. Now we have to make sure the resource constraints are the same. Note that in the TDCE, the government spends all of its revenue each period on government expenditure.

$$p_t g_t + T_t = p_t \tau_{ct} + p_t \tau_{xt} x_t + w_t \tau_{nt} n_t + r_t \tau_{kt} k_t$$
$$p_t g_t = w_t \tau n_t + r_t \tau k_t$$
$$p_t g_t = \tau(w_t n_t + r_t k_t)$$

Since the firm has constant returns to scale,

$$\begin{aligned} F(k_t, n_t) &= F_n(t)n_t + F_k(t)k_t \\ p_t F(k_t, n_t) &= w_t n_t + r_t k_t \end{aligned}$$

Therefore, we know

$$\begin{aligned} p_t g_t &= \tau p_t F(k_t, n_t) \\ g_t &= \tau F(k_t, n_t) \end{aligned}$$

Now, we can substitute this expression for g_t into the resource constraint from (9).

$$\begin{aligned} c_t + k_{t+1} + g_t &= F(k_t, n_t) + (1-\delta)k_t \\ c_t + k_{t+1} + \tau F(k_t, n_t) &= F(k_t, n_t) + (1-\delta)k_t \\ c_t + k_{t+1} &= (1-\tau)F(k_t, n_t) + (1-\delta)k_t \\ c_t + k_{t+1} &= \hat{F}(k_t, n_t) + (1-\delta)k_t \end{aligned}$$

Thus, (9) and (14) are equivalent, and we are done!